Caringbah High School

2016
Year 12 Trial HSC
Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen.
- Board-approved calculators may be used.
- A reference sheet with standard formulae and integrals is provided.
- In Questions 11–16, show relevant mathematical reasoning and/or calculations.

Total marks – 100

Section I Pages 3–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–13

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Given that $z = 5 + 2i$ and $w = i - 3$, what is the value of $2\overline{w} - z$?

(A) $-11 - 4i$

(B) -11

(C) $-1 - 4i$

(D) -1

2. The equation $x^2 - y^2 - 5xy + 5 = 0$ defines y implicity as a function of x.

What is the value of $\frac{dy}{dx}$ at the point $(1,1)$?

(A) -1

(B) $\frac{-3}{7}$

(C) $\frac{3}{7}$

(D) $\frac{7}{3}$

3. What is the eccentricity of the conic described by the equation $9x^2 - 4y^2 = 1$?

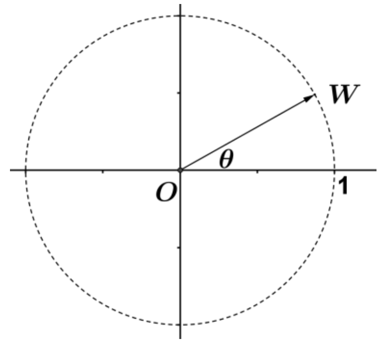
(A) $\sqrt{\frac{5}{2}}$

(B) $\sqrt{\frac{5}{3}}$

(C) $\frac{\sqrt{13}}{2}$

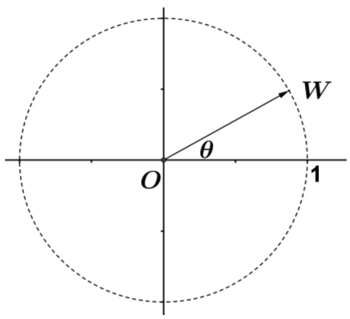
(D) $\frac{\sqrt{13}}{3}$

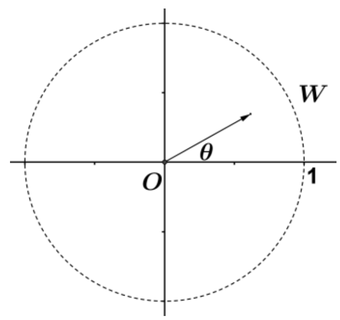
4. $Z = x + iy$ is a complex number, represented on the Argand diagram as shown.

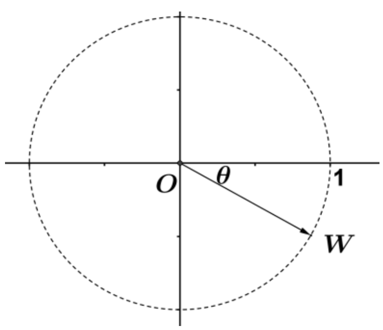


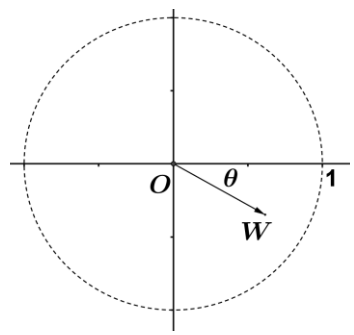
$|Z| = 1$.

Which of the following diagrams would represent the complex number $W = \frac{1}{\bar{Z}}$?

(A) 

(B) 

(C) 

(D) 

5. The equation $4x^3 - 4x^2 - 15x + 18 = 0$ has a double root at $x = \alpha$.
The value of α is

(A) $\alpha = \frac{-3}{2}$ (B) $\alpha = \frac{3}{2}$

(C) $\alpha = \frac{5}{6}$ (D) $\alpha = \frac{-5}{6}$

6. The conic described by the equation $\frac{x^2}{169} + \frac{y^2}{25} = 1$ has directrices

(A) $x = \pm 12$ (B) $x = \pm\frac{169}{12}$

(C) $y = \pm 12$ (D) $y = \pm\frac{169}{12}$

7. If $z = 2 - \sqrt{12}i$, find the argument of z^5.

(A) $\frac{-\pi}{3}$ (B) $\frac{-5\pi}{6}$

(C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{6}$

8. The complex number z satisfies $\arg\left(\frac{z-2}{z+2i}\right) = \frac{-\pi}{2}$.

Find the maximum value of $|z|$.

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

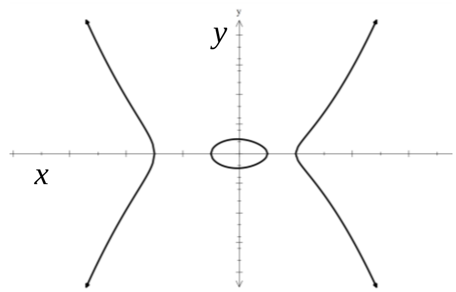
(C) $2 - \sqrt{2}$ (D) $2 + \sqrt{2}$

9. Which of the following is an equivalent expression for $\int \frac{dx}{\sqrt{5 - 4x - x^2}}$?

(A) $\sin^{-1}(x+2) + C$ (B) $\sin^{-1}\left(\frac{x+2}{3}\right) + C$

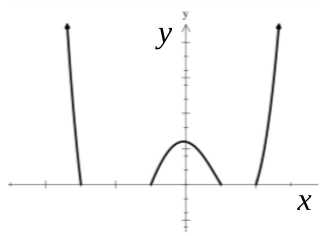
(C) $\sin^{-1}(x-2) + C$ (D) $\sin^{-1}\left(\frac{x-2}{3}\right) + C$

10. $P(x)$ is a polynomial. The graph of $y^2 = P(x)$ is shown below.

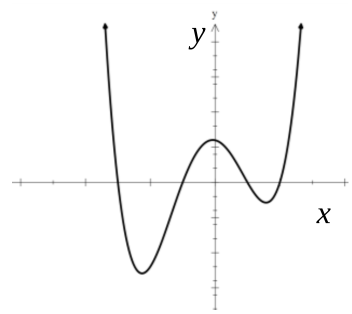


Which of the following graphs is the best representation of $y = P(x)$?

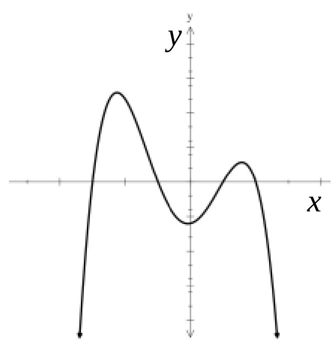
(A)



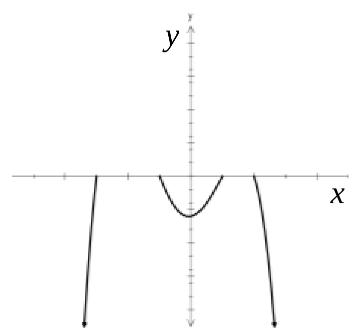
(B)



(C)



(D)



Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) **Marks**

a) If $z = 1 - i$, find z^{-6} in the form $x + iy$. **2**

b) If $z = x + iy$, shade on the Argand diagram the region defined by $z\bar{z} \leq 4$. **2**

c) Evaluate $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$ **3**

d) Show that $\int_{e}^{e^4} \ln x \, dx = 3e^4$ **3**

e) i) Show that $(z - 2i)$ is a factor of $P(z) = z^4 + z^3 + z^2 + 4z - 12$. **1**

ii) Hence, find all zeros of P(z). **1**

f) Let $A = 3 + 4i$ and $B = 9 + 4i$ be two points on the Argand diagram.

i) Sketch the locus defined by $|z - A| = 5$ **1**

ii) Draw a clear sketch of the curve defined by $|z - A| + |z - B| = 12$ **2**

End of Question 11

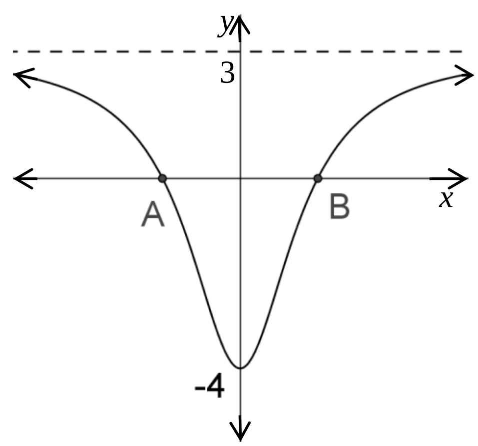
Question 12 (15 Marks) **Marks**

a) i) Express $3\sin\theta + 4\cos\theta$ in the form $r\sin(\theta + \alpha)$, where r and α are constants and α is in radians correct to 3 decimal places. **2**

ii) Hence, or otherwise, show that a particle whose displacement x metres, after t seconds, given by $x = 8\cos^2 t + 6\sin t\cos t - 4$ is moving in simple harmonic motion. **2**

You may assume that $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

b) The graph of $y = f(x)$ is shown.



On separate diagrams, show the following graphs, clearly indicating important features.

i) $y = \dfrac{1}{f(x)}$ **2**

ii) $y = [f(x)]^2$ **2**

iii) $y = \log_e(f(x))$ **2**

Question 12 continues on page 9

Question 12 (continued)

c) The complex number $z = x + iy$, with x and y real, satisfies $|z - i| = Im(z)$.

i) Show that the locus of the point P, representing z, has the Cartesian equation $y = \dfrac{x^2 + 1}{2}$ **2**

ii) By finding the gradients of the tangents to this curve that pass through the origin, state the set of possible values for the principal argument of z. **3**

End of Question 12

Question 13 (15 Marks) **Marks**

a) The base of a solid is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. **4**

Cross-sections perpendicular to the x-axis are right isosceles triangles with one of the equal sides in the base of the solid.

Show that the volume of the solid is $\dfrac{8ab^2}{3}$ units^3.

b) Sketch the curve $y = x\ln(x)$, showing any turning points. **2**

c) i) If z is a complex number defined by $z = \cos\theta + i\sin\theta$, **1**

show that $\dfrac{dz}{d\theta} = iz$

ii) By integrating $\dfrac{dz}{d\theta} = iz$ with respect to z and θ, **2**

show that z can be written in the form $z = e^{i\theta}$.

d) i) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ using the substitution $t = \tan\dfrac{x}{2}$. **3**

ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$ **3**

using the substitution $u = \dfrac{\pi}{2} - x$.

End of Question 13

Question 14 (15 Marks) **Marks**

a) Sketch on the Argand diagram, the locus defined by $\arg(z - 1 + i) = \frac{\pi}{4}$ **1**

b) Two points, $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, lie on the rectangular hyperbola $xy = c^2$.

i) Show that the equation of the tangent at P is $x + p^2y = 2cp$. **2**

ii) The tangents at P and Q meet in T. Find the coordinates of T in terms of c, p and q. **2**

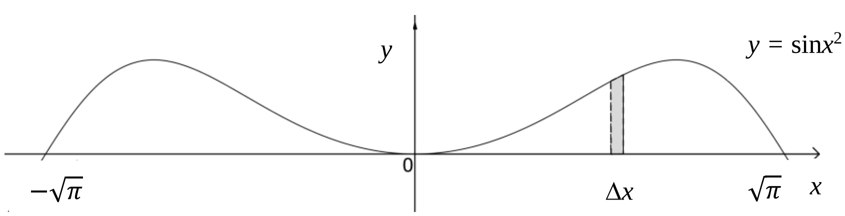
iii) The point T lies on another hyperbola $xy = k^2$, for all positions of P and Q. **1**

Show that $\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$

c) i) Let $I_n = \int_1^e x(\ln x)^n dx$. Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, $n = 1, 2, 3, \ldots$ **3**

ii) Hence evaluate $\int_1^e x(\ln x)^2 dx$ **2**

d) The area bounded by the curve $y = \sin x^2$ and the x-axis, in the domain $-\sqrt{\pi} \le x \le \sqrt{\pi}$, is rotated about the y-axis. **4**



By using the method of cylindrical shells, find the volume of the solid formed.

End of Question 14

Question 15 (15 Marks) **Marks**

a) α, β and γ are roots of the cubic equation $x^3 + mx + n = 0$.

i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ in terms of m and n. **2**

ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ in terms of m and n. **2**

iii) Determine the cubic equation with roots α^2, β^2 and γ^2. **2**

b) For the ellipse $E : 4x^2 + 9y^2 = 36$.

i) Sketch the ellipse *E* indicating the position of its foci *S* and *S'*, and draw in it directrices. **3**

ii) Show that the point $P(3\cos\theta, 2\sin\theta)$ lies on *E*. **1**

iii) Derive the equation of the tangent to *E* at the point *P*. **2**

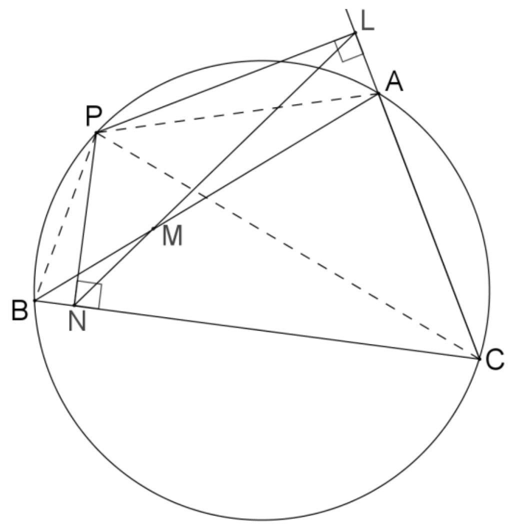
iv) Find the coordinates of the point *Q* where the tangent cuts the major axis. **1**

v) The equation of the normal at *P* is $\frac{3x}{\cos\theta} - \frac{2y}{\sin\theta} = 5$ and it cuts the major axis at *R*. A line parallel to the *y*-axis through *P* cuts the *x*-axis at *T*. Show that $OQ \times RT$ is constant for all positions of *P*. **2**

End of Question 15

Question 16 (15 Marks) **Marks**

a)



ABC is an acute-angled triangle inscribed in a circle. *P* is a point on the minor arc *AB* of the circle. *PL* and *PN* are the perpendiculars from *P* to *CA* produce and *CB* respectively. *LN* cuts *AB* at *M*.

i) Copy the diagram in to your answer booklet. (⅓ page)

ii) Explain why *PNCL* is a cyclic quadrilateral. **1**

iii) Hence show *PBNM* is also a cyclic quadrilateral. **3**

iv) Hence show that *PM* is perpendicular to *AB*. **2**

b) i) Write expressions for $\cos\frac{2\pi}{n}$ and $\sin\frac{2\pi}{n}$ in terms of $\cos\frac{\pi}{n}$ and $\sin\frac{\pi}{n}$ **2**

ii) By using De Moivre's theorem, show that **3**

$$\left(1+\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^n = -2^n\left(\cos\frac{\pi}{n}\right)^n$$

c) i) If a and b are both positive numbers, prove that $a+b \geq 2\sqrt{ab}$. **1**

Hence, or otherwise, prove that when *a*, *b* and *c* are all positive numbers and *a* + *b* + *c* =1, then **3**

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$$

End of Examination

CHS 2016 TRIAL HSC
MATHEMATICS EXT2.

1) A 2) B 3) C 4) A 5) B

6) B 7) C 8) B 9) B 10) B

Question 11

a) $z = 1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

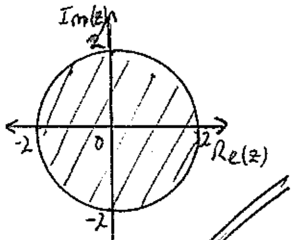
$z^{-6} = (\sqrt{2})^{-6}\,\text{cis}\left(\frac{6\pi}{4}\right)$

$= \frac{1}{8}\,\text{cis}\left(-\frac{\pi}{2}\right)$

$= -\frac{i}{8}$

b) $(x+iy)(x-iy) = x^2+y^2$

$z.\bar{z} \leq 4$



c) $\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$

$= \frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{\pi}{8}$

d) $\int_{e}^{e^4} \ln x\,dx$ $u = \ln x$ $dv = dx$ $du = \frac{1}{x}dx$ $v = x$.

$= x\ln x \Big|_{e}^{e^4} - \int_{e}^{e^4} x \times \frac{1}{x}\,dx$

$= \left[e^4 \times \ln e^4 - e \times \ln e\right] - \left[x\right]_{e}^{e^4}$

$= 4e^4 - e - e^4 + e$

$= 3e^4$

e) i) $P(2i) = (2i)^4 + (2i)^3 + (2i)^2 + 4\times 2i - 12$

$= 16 - 8i - 4 + 8i - 12$

$= 0$

$\therefore (z-2i)$ is a factor of $P(z)$.

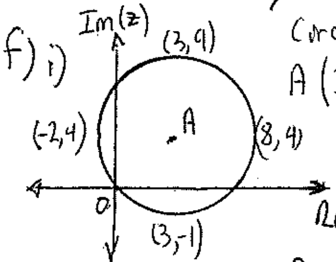
ii) $P(z)$ has real coefficients $\therefore (z+2i)$ is a factor

$\therefore (z+2i)(z-2i) = (z^2+4)$ is a factor.

$z^4 + z^3 + z^2 + 4z - 12 = (z^2+4)(z^2+z-3)$

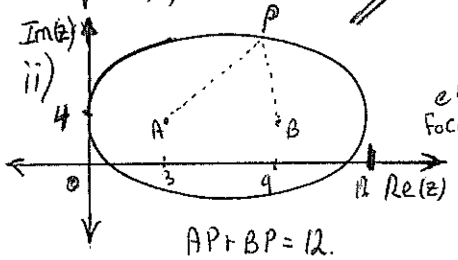
roots are $\pm 2i$, $\frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1\pm\sqrt{13}}{2}$

f) i)



Circle
$A(3,4)$ = centre
radius = 5

ii)



ellipse
Focii $A(3,4)$
$B(9,4)$
$a = 6$

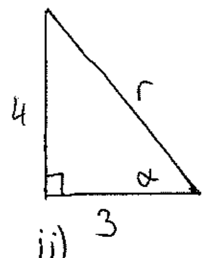
$AP + BP = 12$.

Question 12

a) i) $3\sin\theta + 4\cos\theta$.

$= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$.

$r\cos\alpha = 3$ $r\sin\alpha = 4$



$r = 5$ $\tan\alpha = \frac{4}{3}$

$\alpha \doteq 0.927$

$3\sin\theta + 4\cos\theta = 5\sin(\theta + 0.927)$

ii) $x = 8\cos^2 t + 6\sin t\cos t - 4$

$= 8\cos^2 t + 3\sin 2t - 4$

$\dot{x} = -16\cos t\sin t + 6\cos 2t$

$= -8\sin 2t + 6\cos 2t$

$\ddot{x} = -16\cos 2t - 12\sin 2t$

$= -16(2\cos^2 t - 1) - 24\sin t\cos t$

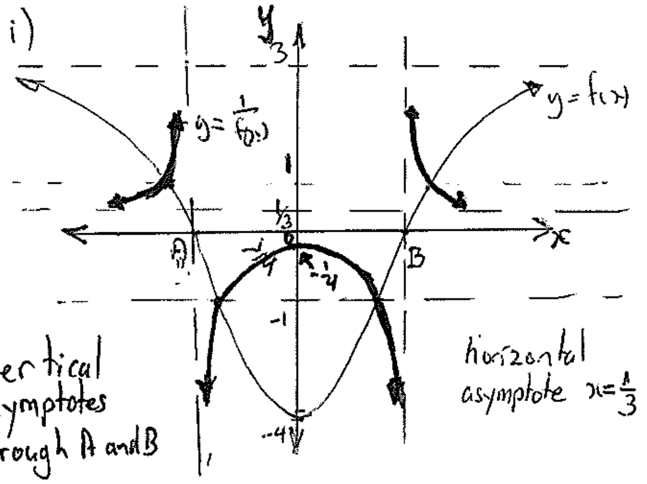
$= -32\cos^2 t - 24\sin t\cos t + 16$

$= -4(8\cos^2 t + 6\sin t\cos t - 4)$

$= -4x$

$\therefore$ SHM, $n = 2$, $T = \pi$

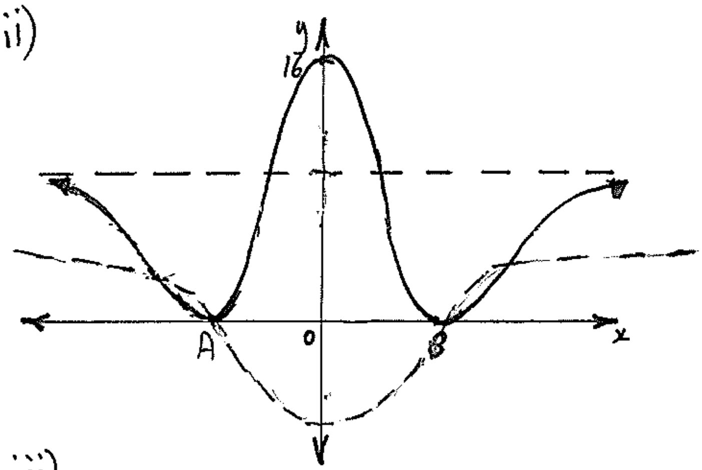
12b) i)



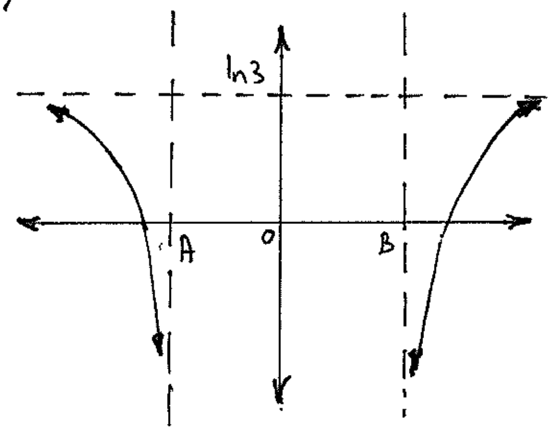
vertical asymptotes through A and B

horizontal asymptote $x=\frac{1}{3}$

ii)



iii)



c) i) $|z-i| = \text{Im}\, z$

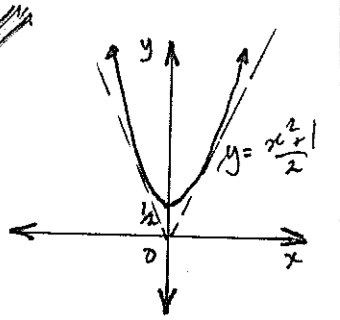
$$\sqrt{(x^2+(y-1)^2)} = y$$

$$x^2+y^2-2y+1 = y^2$$

$$x^2-2y+1=0$$

$$2y = x^2+1$$

$$y = \frac{x^2+1}{2}.$$



ii) $y = \frac{x^2}{2}+\frac{1}{2}$

$\frac{dy}{dx} = x$ If it passes through 0, then gradient $= \frac{y}{x}$

$$\frac{y}{x} = x$$

$$y = x^2$$

$$\frac{x^2+1}{2} = x^2$$

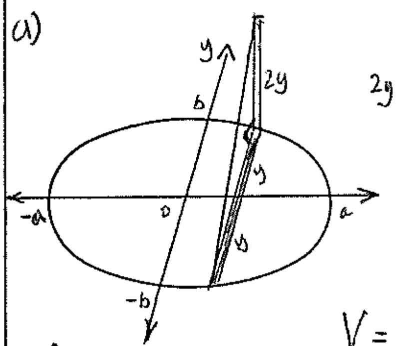
$$x^2 = 1 \quad \therefore x = \pm 1,\ y=1$$

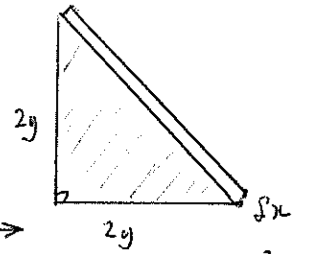
ie tangents inclined at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

$$\therefore \frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$$

Question 13

a)





$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\delta V = \frac{1}{2}\times(2y)^2\delta x.$$

$$V = 2\int_0^a 2y^2\,dx.$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2}(a^2-x^2)$$

$$V = \frac{4b^2}{a^2}\int_0^a (a^2-x^2)\,dx$$

$$= \frac{4b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{4b^2}{a^2}\left[\left(a^3-\frac{a^3}{3}\right)-0\right]$$

$$= \frac{4b^2}{a^2}\times\frac{2a^3}{3}$$

$$= \frac{8ab^2}{3}$$

b) $y = x\ln x \Rightarrow x>0$

$$y' = x\times\frac{1}{x}+\ln x.$$

$$= 1+\ln x$$

$y'=0$ $\ln x = -1$

$$x = e^{-1}$$

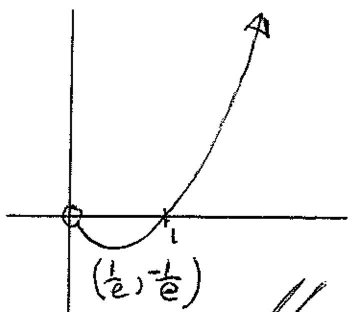
$$= \frac{1}{e}$$

$$y = -\frac{1}{e}$$

$y''=\frac{1}{x}$ $\therefore$ min TP at $\left(\frac{1}{e},-\frac{1}{e}\right)$

as $x>0 \Rightarrow y''>0$

ie always concave up.



13 c) i) $z = \cos\theta + i\sin\theta$

$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$iz = i(\cos\theta + i\sin\theta)$

$= i\cos\theta - \sin\theta$

$= \frac{dz}{d\theta}$

ii) $\frac{dz}{d\theta} = iz$

$\int \frac{dz}{z} = \int i\, d\theta$

$\ln z = i\theta + c$

$\theta = 0 \quad z = \cos 0 + i\sin 0 = 1$

so $\ln 1 = 0 + c \quad \therefore c = 0$

$\ln z = i\theta$

$z = e^{i\theta}$ [A useful extension to complex numbers!]

d) i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$

| $t = \tan\frac{x}{2}$
| $\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$
| $x = 2\tan^{-1}t$
| $dx = \frac{2\,dt}{1+t^2}$
| $x = 0 \quad t = 0$
| $x = \frac{\pi}{2} \quad t = 1$

$\int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \times \left(\frac{1+t^2}{1+t^2}\right)$

$\int_0^1 \frac{2\,dt}{1+t^2+1-t^2+2t}$

$\int_0^1 \frac{2\,dt}{2+2t}$

$\int_0^1 \frac{dt}{1+t} = \left[\ln|1+t|\right]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$

ii) $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x} = \int_{\frac{\pi}{2}}^{0} \frac{(\frac{\pi}{2}-u)(-du)}{1+\cos(\frac{\pi}{2}-u)+\sin(\frac{\pi}{2}-u)}$

$u = \frac{\pi}{2} - x$

$du = -dx$

$x = 0 \quad u = \frac{\pi}{2}$

$x = \frac{\pi}{2} \quad u = 0$

$= \int_{\frac{\pi}{2}}^{0} \frac{-\frac{\pi}{2}\,du}{1+\sin u+\cos u} - \int_{\frac{\pi}{2}}^{0} \frac{-u\,du}{1+\cos u+\sin u}$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{du}{1+\cos u+\sin u} - \int_0^{\frac{\pi}{2}} \frac{u}{1+\cos u+\sin u}$ ← original integral

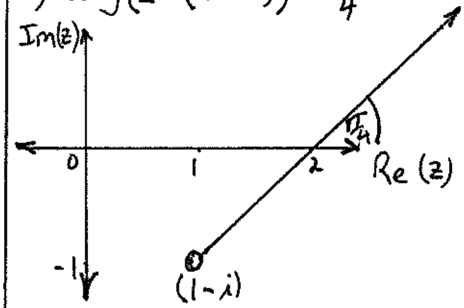
let $u = x$.

$2I = \frac{\pi}{2} \times \ln 2$ [from i)]

$I = \frac{\pi}{4}\ln 2$

Question 14

a) $\arg(z-(1-i)) = \frac{\pi}{4}$



b) i) $y = \frac{c^2}{x} \quad \frac{dy}{dx} = \frac{-c^2}{x^2}$

at $x = cp \quad m = \frac{-c^2}{c^2p^2} = \frac{-1}{p^2}$

Eqn of tangent

$(y - \frac{c}{p}) = \frac{-1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$

ii) $x + p^2y = 2cp$
$x + q^2y = 2cq$ Eqns of tangents

$y(p^2-q^2) = 2c(p-q)$

$y(p-q)(p+q) = 2c(p-q)$

As $(p-q) \neq 0$

$y(p+q) = 2c$

$y = \frac{2c}{p+q}$

$x = 2cp - p^2y$

$= 2cp - \frac{2cp^2}{p+q}$

$= \frac{2cp^2+2cpq}{p+q} - \frac{2cp^2}{p+q}$

$= \frac{2cpq}{p+q}$ coord of T $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

iii) $xy = k^2 \Rightarrow \frac{2cpq}{(p+q)} \times \frac{2c}{(p+q)} = k^2$

$\frac{4c^2pq}{(p+q)^2} = k^2$

$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$

14c)

i) $I_n = \int_1^e x(\ln x)^n \, dx$ $\quad u = (\ln x)^n \quad dv = x\,dx$

$du = n(\ln x)^{n-1} \times \frac{1}{x}\,dx \quad v = \frac{x^2}{2}$

$$I_n = \frac{x^2}{2}(\ln x)^n \Big|_1^e - \int_1^e \frac{n x^2}{2}(\ln x)^{n-1} \times \frac{1}{x}\,dx$$

$$= \left[\frac{e^2}{2}(\ln e)^n - \frac{1}{2}(\ln 1)^n\right] - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$$

$\ln e = 1 \quad \ln 1 = 0$

$$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

ii) $n = 2$

$I_2 = \frac{e^2}{2} - I_1$

$n = 1$

$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0$

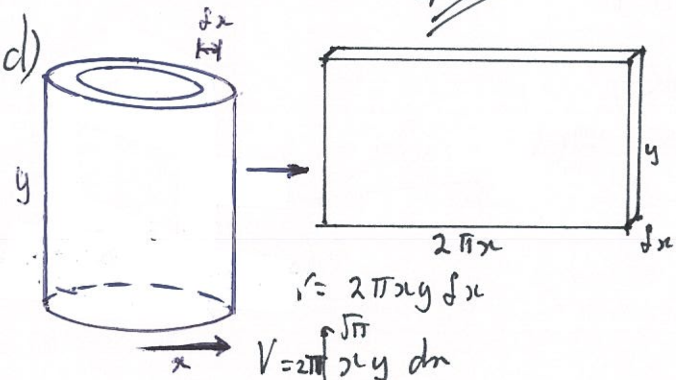
$I_0 = \int_1^e x(\ln x)^0\,dx = \left[\frac{1}{2}x^2\right]_1^e$

$= \frac{e^2}{2} - \frac{1}{2}$

$$\therefore I_2 = \frac{e^2}{2} - \left[\frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{e^2}{4} - \frac{1}{4} \text{ or } \frac{e^2 - 1}{4}$$

d)



$V = 2\pi x y\,\delta x$

$$V = 2\pi \int_0^{\sqrt{\pi}} x y\,dx$$

$$= 2\pi \int_0^{\sqrt{\pi}} x \sin x^2\,dx$$

let $u = x^2 \quad du = 2x\,dx$

$x = 0 \quad u = 0$

$x = \sqrt{\pi} \quad u = \pi$

$$V = \pi \int_0^{\pi} \sin u\,du$$

$$= \pi\left[-\cos u\right]_0^{\pi}$$

$$= \pi[1 + 1]$$

$= 2\pi \ u^3$

Question 15

a) i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$

$x^3 + 0x^2 + mx + n = 0$

$-\frac{b}{a} = 0, \quad \frac{c}{a} = m \quad -\frac{d}{a} = -n$

So $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-m}{n}$

ii) $\alpha^3 = -m\alpha - n$

$\beta^3 = -m\beta - n$

$\gamma^3 = -m\gamma - n$

$\alpha^3 + \beta^3 + \gamma^3 = -m(\alpha + \beta + \gamma) - 3n$

$\alpha + \beta + \gamma = 0$

$= -3n$

iii) Sub $x = \sqrt{x}$.

$(\sqrt{x})^3 + m\sqrt{x} + n = 0$

$x\sqrt{x} + m\sqrt{x} = -n$

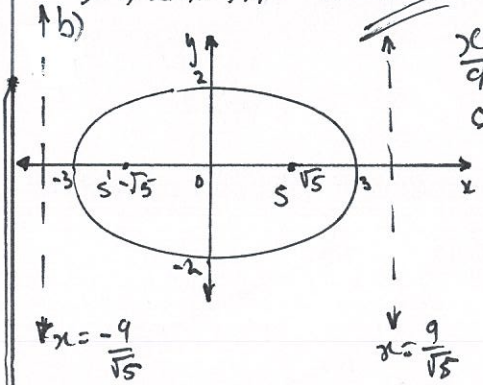
$\sqrt{x}(x + m) = -n$

squaring both sides

$x(x + m)^2 = n^2$

$x^3 + 2mx^2 + m^2x - n^2 = 0$

b)



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$a = 3 \quad b = 2$

$b^2 = a^2(1 - e^2)$

$\frac{4}{9} = 1 - e^2$

$e = \frac{\sqrt{5}}{3}$

ii) $\frac{(3\cos\theta)^2}{9} + \frac{(2\sin\theta)^2}{4} = \frac{9\cos^2\theta}{9} + \frac{4\sin^2\theta}{4}$

$= \cos^2\theta + \sin^2\theta$

$= 1$

$\therefore P$ lies on E.

iii) $4x^2 + 9y^2 = 36$

$8x + 18y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-8x}{18y}$

$= \frac{-4x}{9y}$

at E $m = \frac{-4(3\cos\theta)}{9(2\sin\theta)}$

$m = -\frac{2}{3}\frac{\cos\theta}{\sin\theta}$

Eqn of tangent

$(y - 2\sin\theta) = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

$3\sin\theta\, y - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

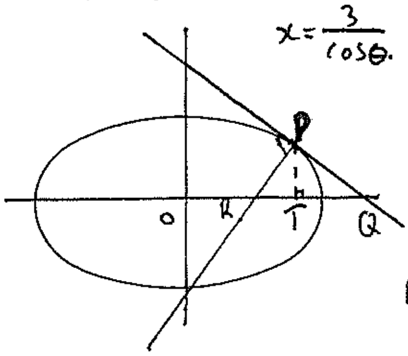
$2\cos\theta\, x + 3\sin\theta\, y = 6(\sin^2\theta + \cos^2\theta)$

$= 6$

$\frac{\cos\theta\, x}{3} + \frac{\sin\theta\, y}{2} = 1$

15b iv) $y=0$ $2\cos\theta\, x=6$

$x=\frac{3}{\cos\theta}$



$OQ=\frac{3}{\cos\theta}$

$OT = 3\cos\theta$

$OR = \frac{5\cos\theta}{3}$

$RT = OT - OR$

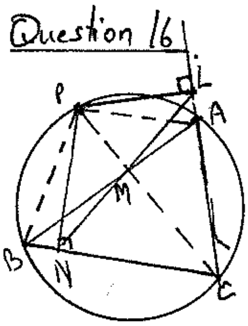
$= 3\cos\theta - \frac{5\cos\theta}{3}$

$= \frac{4\cos\theta}{3}$

$OQ \times RT = \frac{3}{\cos\theta} \times \frac{4\cos\theta}{3}$

$= 4$ ie constant for all positions of P.

Question 16



ii) $\angle PLC = \angle PNC = 90°$ (perpendiculars)

$\therefore$ opp angles of quadrilateral are supplementary.

$\therefore$ PNCL is a cyclic quad.

iii) As PNCL are concyclic (from i))

$\angle PNL = \angle PCL$ angles at circumference standing on chord PL

APBC are concyclic so

$\angle PBA = \angle PCA$ angles at circum. standing on chord PA

$\angle PCL$ is also $\angle PCA$

so $\angle PNL = \angle PCL = \angle PCA = \angle PBA$

or $\angle PNL = \angle PBA$

$\angle PNL$ can be named $\angle PNM$

$\angle PBA$ can be named $\angle PBM$

so $\angle PNM = \angle PBM$

$\therefore$ PBNM is a cyclic quad

equal angles at circumference on chord PM.

iii) $\angle BNP$ and $\angle BMP$ stand on chord BP in circle PBNM (from ii).

$\therefore \angle BNP = \angle BMP$

as $PN \perp BC$ (given.

$\angle PMB = 90°$

ie $PM \perp AB$.

b) i) $\cos\frac{2\pi}{n} = \cos^2\frac{\pi}{n} - \sin^2\frac{\pi}{n}$

$\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

b ii)

$1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = \left(\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}\right) + \left(\cos^2\frac{\pi}{n} - \sin^2\frac{\pi}{n}\right) + \left(2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)$

$= 2\cos^2\frac{\pi}{n} + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

$= 2\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)$

$\left(1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = 2^n\left(\cos\frac{\pi}{n}\right)^n\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^n$

$= 2^n\left(\cos\frac{\pi}{n}\right)^n\left(\cos\pi + i\sin\pi\right)$

$\text{cis}\,\pi = -1$

$= -2^n\left(\cos\frac{\pi}{n}\right)^n$

c) i) $(a-b)^2 \geq 0$

$(a-b)^2 = a^2+b^2-2ab \geq 0$

$(a+b)^2 - 4ab \geq 0$

$(a+b)^2 \geq 4ab$

$a+b \geq 2\sqrt{ab}$

(positive case as a, b are both positive)

ii)

$\left(\frac{1}{a}-1\right) = \frac{1-a}{a}$ If $a+b+c=1$

$a = 1-b-c$

$= \frac{1-(1-b-c)}{a}$

$= \frac{b+c}{a}$

$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \frac{(b+c)}{a}\frac{(a+c)}{b}\frac{(a+b)}{c}$

from i) $a+b \geq 2\sqrt{ab}$

$\geq \frac{2\sqrt{bc}}{a} \times \frac{2\sqrt{ac}}{b} \times \frac{2\sqrt{ab}}{c}$

$\geq \frac{8\sqrt{a^2b^2c^2}}{abc}$

$\geq \frac{8abc}{abc}$

≥ 8